NPS-MA-94-002

NAVAL POSTGRADUATE SCHOOL

Monterey, California

MEASURES OF CHANGE AND THE DETERMINATION OF EQUIVALENT CHANGE

by

C.L. Frenzen

Technical Report For Period
April 1994 - June 1994

Approved for public release; distribution unlimited

d for: Naval Postgraduate School
Monterey, CA 93943

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NPS-MA-94-002

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REPORT DOCUMENTATION PAGE

Form Approved
OMB No 0704-0188

1a REPORT SECURITY CLASSIFICATION: Unclassified	1b RESTRICTIVE MARKINGS
2a SECURITY CLASSIFICATION AUTHORITY	3 DISTRIBUTION / AVAILABILITY OF REPORT: Approved for public release; distribution unlimited
2b DECLASSIFICATION / DOWNGRADING SCHEDULE	
4 PERFORMING ORGANIZATION REPORT NUMBER(S): NPS-MA-94-002	5 MONITORING ORGANIZATION REPORT NUMBER(S): NPS-MA-94-002
6a NAME OF PERFORMING ORGANIZATION: Naval Postgraduate School; 6b OFFICE SYMBOL (If applicable): MA	7a NAME OF MONITORING ORGANIZATION: Naval Postgraduate School
6c ADDRESS (City, State, and ZIP Code): Monterey, CA 93943	7b ADDRESS (City, State, and ZIP Code): Monterey, CA 93943
8a NAME OF FUNDING / SPONSORING ORGANIZATION: Naval Postgraduate School; 8b OFFICE SYMBOL (If applicable): MA	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: OM,N
8c ADDRESS (City, State, and ZIP Code): Monterey, CA 93943	10 SOURCE OF FUNDING NUMBERS: PROGRAM ELEMENT NO; PROJECT NO; TASK NO; WORK UNIT ACCESSION NO

11 TITLE (Include Security Classification)
Measures of Change and the Determination of Equivalent Change

12 PERSONAL AUTHOR(S)
C.L. Frenzen

13a TYPE OF REPORT	13b TIME COVERED	14 DATE OF REPORT (Year, Month, Day)	15 PAGE COUNT
Technical	FROM 4-94 TO 6-94	6 June 94	28

16 SUPPLEMENTARY NOTATION

17 COSATI CODES			18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB GROUP	Equivalent change

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

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20 DISTRIBUTION / AVAILABILITY OF ABSTRACT	21 ABSTRACT SECURITY CLASSIFICATION	
☐ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	Unclassified	
22a NAME OF RESPONSIBLE INDIVIDUAL: C.L. Frenzen	22b TELEPHONE (Include Area Code): 408-656-2435	22c OFFICE SYMBOL: MA/Fr

DD Form 1473, JUN 86 — Previous editions are obsolete — SECURITY CLASSIFICATION OF THIS PAGE

S/N 0102-LF-014-6603

Measures of Change and the Determination of Equivalent Change *

C.L. Frenzen
Naval Postgraduate School
Monterey, CA 93943-5100, U.S.A.

Abstract

Equivalent change in percentages, probabilities, or other variables belonging to a finite interval cannot be properly determined using methods appropriate for the real or positive real numbers, since these may require a variable to fall outside its interval of definition. A general theory for determining equivalent change on any open interval G of real numbers is developed. Properties for measures of change are proposed which give G a group structure order isomorphic to the naturally ordered additive group of real numbers. Different group operations on G determine numerically different measures of change, and numerically different results for equivalent change. Requiring the group product on G to be a rational function of its factors yields familiar results for equivalent change on the real and positive real numbers, and a function recently proposed by Ng when G is the open unit interval. Ng's function is not uniquely characterized by his twelve 'reasonable' properties, but is uniquely determined when the group product on G depends rationally on its factors. Geometrical interpretations of these results for the real numbers, positive real numbers, and the open unit interval are also given.

1. Introduction. The measurement of change and the determination of equivalent change are common to many areas of human endeavor. Consider

*1991 *Mathematics Subject Classification.* Primary 60A05, 20A05

three examples:

[1] The balance in a bank account changes from −\$200 to \$150. A second account contains \$150. What new balance in the second account gives a change in its balance equivalent to the change in the first account?

[2] The enrollment in a course changes from 30 students to 20 students. A second course has 9 students. What new enrollment in the second course gives a change in its enrollment equivalent to the change in the first course?

[3] A teacher adjusts the test score of a student from 50% to 75%. A second student has a score of 80%. What new score for the second test gives a change in its score equivalent to the change in the first test?

The following question is common to each of these examples:

[Q] Variable x changes from x_1 to x_2. Given y_1, what new value y_2 of y gives a change in variable y equivalent to the change in variable x?

Common answers to it are obtained from

$$y_2 - y_1 = x_2 - x_1 \quad \text{(equal differences)} \tag{1}$$

and

$$\frac{y_2}{y_1} = \frac{x_2}{x_1} \quad \text{(equal proportions)}. \tag{2}$$

They work best when x and y are real numbers or positive real numbers respectively. Equations (1) and (2) are what most people apply to examples [1] and [2] to obtain the answers \$500 and 6 students. Answering question [Q] for example [3] is not quite as straightforward for when variables x and y belong to a finite interval, as percentages and probabilities do, (1) and (2) no longer yield valid answers. To see this, suppose x and y represent the first and second test scores respectively in example [3]. Then $x_1 = 50\%$, $x_2 = 75\%$, and $y_1 = 80\%$. 'Equal differences' gives $y_2 = 105\%$, while 'equal proportions' gives $y_2 = 120\%$. Both answers are unacceptable since the score y cannot exceed 100% without falling outside its interval of definition.

Surprisingly, no agreed upon method for determining equivalent change in percentages and probabilities appears to exist. Such a method would answer

question [Q] on the unit interval by expressing y_2 as a function of x_1, x_2, and y_1, in the same way that (1) and (2) give

$$y_2 = x_2 - x_1 + y_1 \tag{3}$$

and

$$y_2 = \frac{x_2}{x_1} y_1 \tag{4}$$

for the real and positive real numbers respectively. Ng [1] gave twelve 'reasonable' properties for such functions when variables x and y belong to the unit interval. He gave the specific function

$$y_2 = \frac{x_2 y_1 (1 - x_1)}{x_1(1 - x_2) + y_1(x_2 - x_1)} \tag{5}$$

as an example satisfying his properties and claimed that it was, in some unspecified sense, the most natural answer to question [Q] in this case. Ng also raised the question of uniqueness: to what extent do his properties uniquely characterize the answer in (5)? No definite conclusions about uniqueness were reached though, and no reasons were advanced for calling (5) the most natural answer to question [Q]. Ng, however, stated his confidence that (5) "provides one (if not the only) acceptable function for equivalent changes that may lead to widespread practical application in many fields." (Ng [1], p. 300) The purpose of this paper is to develop a general theory connecting the answers to question [Q] in (3), (4), and (5), to explain what the word 'natural' means, and to show that Ng's properties do not uniquely characterize the answer in (5).

We shall assume that variables x and y belong to an open interval G of the real numbers $\mathbf{R}$, perhaps infinite. Equations (3), (4), and (5) are then possible answers to question [Q] when $G = \mathbf{R}$, $\mathbf{R}^{\text{pos}}$, and (0,1) respectively, where $\mathbf{R}^{\text{pos}}$ is the set of positive real numbers and (0,1) is the open unit interval. To see how an answer to question [Q] is determined, note, that each side of (1),for example, is a measure of the change occurring in the variable appearing on that side. Equivalent change in x and y occurs when these measures of change are equal. Solving for y_2 in terms of x_1, x_2, and y_1 then gives the answer to question [Q] appearing in (3). A similar pattern occurs in going from (2) to (4). The process of obtaining these answers begins with the notion of a measure of change defined on the set G.

In the next section we propose several general properties for measures of change on the open interval G; these properties are sufficient to give G an ordered group structure which is isomorphic to $(\mathbf{R}, +)$, the naturally ordered additive group of real numbers. Group operations on G allow us to define a measure of change on G. Since there are many possible group operations on G which give it a group structure order isomorphic to $(\mathbf{R}, +)$, there are many possible measures of change on G and a unique answer to question [Q] does not exist.

We shall denote by $(G, *)$ a group defined on G with group product $*$. Perhaps the 'simplest' conceivable group product on G is one which depends *rationally* on its factors. In section 3 we show that a one-parameter family of groups exists on G, each member of which has a group product depending *rationally* on its factors. Every member of this family of groups is order isomorphic to $(\mathbf{R}, +)$ and can be denoted by $(G, *_{e_\circ})$ where $e_\circ \in G$ is the identity element of the group $(G, *_{e_\circ})$ and serves as a parameter indexing the family. Though different values of the parameter $e_\circ \in G$ determine different group products $*_{e_\circ}$ and different measures of change on G, every group in the one-parameter family $\{(G, *_{e_\circ}) : e_\circ \in G\}$ determines the *same* answer to question [Q]. This is the sense in which (5), (4), and (3) are the most natural answers to question [Q] when G = (0,1), $\mathbf{R}^{\text{pos}}$, and $\mathbf{R}$ respectively: to ensure appropriate properties for measuring change on the interval G, a group structure must exist on G which is order isomorphic to $(\mathbf{R}, +)$. The resulting measure of change and answer to question [Q] determined by the group structure on G are simplest when the group product depends rationally on its factors. Under these conditions, (5), (4), and (3) are the *unique* answers to question [Q] when G = (0,1), $\mathbf{R}^{\text{pos}}$, and $\mathbf{R}$ respectively. The same approach yields unique answers to question [Q] when G is an interval of the form $(-\infty, a), (a, b)$, or (a, ∞), where $a, b \in \mathbf{R}$ and $a < b$.

In section 4 the results of sections 2 and 3 are summarized and used to formulate a set of properties leading to a unique answer to question [Q] on any open interval G. Geometric interpretations of the answers in (3), (4), and (5) are also given. In an appendix we examine Ng's twelve properties and the extent to which they constrain answers to question [Q] on the unit interval (0,1). We show that his properties do not lead to a unique answer to question [Q] on (0,1) and give some examples of this nonuniqueness.

The existence of a method to determine equivalent change on G allows one to determine which variable undergoes the greatest change. For example,

using (5), the answer to question [Q] in example [3] is $y_2 = 12/13 \approx 92.3\%$. Thus we can say that when $y_2 < 12/13$ the change in y is less than the change in x, but when $y_2 > 12/13$ the change in y is greater than the change in x However we caution that the answer $y_2 = 12/13$ in this example, like the answers given earlier for examples [1] and [2], is not unique and depends on the group operations chosen for $(G, *)$. As we shall see, $x_1 * x_2$ represents the element of G obtained by changing x_1 by the amount x_2 (or x_2 by the amount x_1). Since G is an open interval of $\mathbf{R}$, for given $x_1, x_2 \in G$ we feel intuitively that the 'simplest' definition of $x_1 * x_2$ should employ only the usual field operations of $\mathbf{R}$—addition, subtraction, multiplication, and division—to form $x_1 * x_2$ from x_1 and x_2. This means that the group product $x_1 * x_2$ depends rationally on its factors. Hence, our intuition impels us to call (3), (4), and (5) the most 'natural' answers to question [Q] when $G = \mathbf{R}$, $\mathbf{R}^{\text{pos}}$, and (0,1) respectively. Indeed, it is difficult to think of answers for examples [1] and [2] other than the 'natural' ones. We emphasize again that the answer to question [Q] on the open interval G is not unique and arises from the specific choice of group operations in $(G, *)$, a group on G which is order isomorphic to $(\mathbf{R}, +)$.

Finally, we remark following Ng [1], that other notions of equivalent change may exist. For example, the age distribution of students in school implies that it may be easy to reduce juvenile illiteracy to as low as 5%, say, while it is substantially more difficult to reduce adult illiteracy below 15%. So, in some sense, it may be reasonable to say that reducing adult illiteracy from 15% to 14% is equivalent to the reduction of juvenile illiteracy from 15% to 5%. But since juvenile and adult illiteracy both start at 15% ($x_1 = y_1 = 15\%$), clearly both must increase or decrease to the same percentage in order to register equivalent changes in a *mathematical* sense. As in Ng [1], it is this mathematical sense of equivalent change which is discussed here, since specific conditions, contingencies, and peculiarities associated with individual variables in particular problems cannot be incorporated into a single theory.

2. Properties of Measures of Change. To answer question [Q] we need a way to represent changes occurring in variables defined on G. Our fundamental assumption is:

[**A**] For every ordered pair of elements $(x_1, x_2) \in \mathrm{G}$ there is a unique element of G representing the change in variable x from x_1 to x_2.

How does one measure the change in variable x from x_1 to x_2? Our approach first changes both x_1 and x_2 by an amount calculated to make the new value of x_1 equal to some chosen reference element of G. A symbolic representation of this process might look like $x_1 \mapsto x_1 * x_1^{-1}$, $x_2 \mapsto x_2 * x_1^{-1}$. The new value of x_1 is $x_1 * x_1^{-1}$, an element of G to eventually represent no change, and the new value of x_2 is $x_2 * x_1^{-1}$. Since x_1 and x_2 each change by the same amount in this process, the measure of change between them remains the same and equals the change from $x_1 * x_1^{-1}$ (the reference element of G) to $x_2 * x_1^{-1}$. We then take the reference element $x_1 * x_1^{-1}$ as origin and define this latter change to be $x_2 * x_1^{-1}$ itself, so that the change in variable x from x_1 to x_2 is the element $x_2 * x_1^{-1}$. But given G, how can the operations $*$ and $(\cdot)^{-1}$ be defined? This question will be answered in (15).

We denote the change in variable x from $x_1 \in \mathrm{G}$ to $x_2 \in \mathrm{G}$ by x_2/x_1. Assumption [A] implies that / is a binary operation on G:

[$\mathbf{P_0}$] For every ordered pair (x_1, x_2) of elements in G, x_2/x_1 is defined so that $x_2/x_1 = z$ is a unique element of G.

Given $x_1 \in \mathrm{G}$, assumption [A] and property [$\mathrm{P_0}$] imply that a unique element $x_1/x_1 \in \mathrm{G}$ exists which represents no change in variable x from x_1 to x_1. We shall require the element x_1/x_1 to represent no change for *every* element in G:

[$\mathbf{P_1}$] For every $x_1, x_2 \in \mathrm{G}$, $x_1/x_1 = x_2/x_2$.

Property [$\mathrm{P_1}$] singles out a unique element $x_1/x_1 \in \mathrm{G}$ to represent no change; we call it the identity element of G. As suggested above, the identity element may be used as a reference element from which to measure change. For $x_2 \in G$, the change from the identity x_1/x_1 to x_2 is $x_2/(x_1/x_1)$. By property [$\mathrm{P_1}$] this change depends only on x_2, so taking the identity element x_1/x_1 as origin, we let x_2 itself represent the change in variable x from x_1/x_1 to x_2:

[$\mathbf{P_2}$] For every $x_1, x_2 \in \mathrm{G}$, $x_2/(x_1/x_1) = x_2$.

For $x_1, x_2 \in G$, the change in variable x from x_2 to x_3 should, in some sense, be the inverse of the change in variable x from x_3 to x_2. Since property [$\mathrm{P_2}$] implies $(x_2/x_3)/(x_1/x_1) = x_2/x_3$, we shall require:

[$\mathbf{P_3}$] For every $x_2, x_3 \in$ G, $(x_1/x_1)/(x_2/x_3) = x_3/x_2$.

These properties suggest a unary operation of inverse $(\cdot)^{-1}$ defined on G by

$$x_1^{-1} = (x_1/x_1)/x_1. \tag{6}$$

If $x_1 \in G$ represents a certain change, x_1^{-1} is the inverse change. By using properties [$\mathrm{P_2}$] and [$\mathrm{P_3}$], we can show that

$$\begin{aligned} (x_1^{-1})^{-1} &= (x_1^{-1}/x_1^{-1})/x_1^{-1} \\ &= (x_1^{-1}/x_1^{-1})/[(x_1/x_1)/x_1] \\ &= x_1/(x_1/x_1) \\ &= x_1. \end{aligned} \tag{7}$$

Expressed in terms of the inverse operation, property [$\mathrm{P_3}$] is $(x_2/x_3)^{-1} = x_3/x_2$.

Next, we define a binary product $*$ on G by

$$x_2 * x_1 = x_2/x_1^{-1} \tag{8}$$

for $x_1, x_2 \in G$. The measure of change in variable x from x_1 to x_2 is then

$$x_2/x_1 = x_2/(x_1^{-1})^{-1} = x_2 * x_1^{-1}, \tag{9}$$

giving the method of measuring change introduced following assumption [A]. Necessary for its success is the assumption that the change from $x_1 \in G$ to $x_2 \in G$ remains invariant when these elements are each combined with x_1 using the binary operation /. In other words, if $x_1 \mapsto x_1/x_1$ and $x_2 \mapsto x_2/x_1$, then $(x_2/x_1)/(x_1/x_1) = x_2/x_1$. This is property [$P_2$]. Additionally, we shall require the measure of change between two elements $x_2, x_3 \in G$ to remain invariant when each is combined with *any* element $x_1 \in$ G using the binary operation / :

[$\mathbf{P_4}$] For all $x_1, x_2, x_3 \in$ G, $(x_2/x_1)/(x_3/x_1) = x_2/x_3$.

Property [$\mathrm{P_4}$] gives the set G and the binary operation / a certain homogeneity with respect to the measurement of change.

Properties [$\mathrm{P_0}$]–[$\mathrm{P_4}$] are sufficient to ensure that the set G with the unary operation of inverse $(\cdot)^{-1}$ and the binary product $*$ is a group; see, for example, Hall [2], p. 6. Whittaker [3] has shown that a group structure on the set G can be inferred from a binary operation / on G satisfying property [$\mathrm{P_0}$] and a stronger form of property [$\mathrm{P_4}$]:

[$\mathbf{P}_4'$] For all $x_1, x_2, y_1, y_2 \in G$, $(x_2/x_1) = (y_2/y_1)$ if, and only if, there is a $z \in G$ such that $y_2 = x_2/z$ and $y_1 = x_1/z$.

Assumption [A] gives the elements of G an additional role as measures of change on G, so the group product $x_1 * x_2$ has two possible interpretations depending on the particular roles given to its factors x_1, x_2. If $x_2 \in G$ and x_1 represents a measure of change, the product $x_1 * x_2$ could be interpreted as the new element of G obtained by changing x_2 by the amount x_1. Under this interpretation (9) implies that the change in variable x from x_2 to $x_1 * x_2$ is

$$(x_1 * x_2)/x_2 = (x_1 * x_2) * x_2^{-1} = x_1 * (x_2 * x_2^{-1}) = x_1, \tag{10}$$

since the product $*$ is associative and $x_2 * x_2^{-1}$ is the identity element of $(G, *)$. However, if $x_1 \in G$ and x_2 represents a measure of change, the group product $x_1 * x_2$ might also be interpreted as the new element of G obtained by changing x_1 by the amount x_2. Under this interpretation we should have

$$(x_1 * x_2)/x_1 = x_1 * x_2 * x_1^{-1} = x_2. \tag{11}$$

Now the second equality in (11) is valid for all $x_1, x_2 \in$ G if and only if the group (G,$*$) is abelian, or commutative. When expressed in terms of the binary operation /, (11) becomes

[$\mathbf{P}_5$] For all $x_1, x_2 \in$ G, $x_2 = x_1/(x_1/x_2)$.

Whittaker [3, p. 637] gives properties [P_0], [P_4], and [P_5] as necessary and sufficient conditions for the group (G, $*$) to be abelian. However the reasons for adopting property [P_5] are not immediate so we will not assume it. The group (G, $*$) does not obviously have to be abelian, but considerations of order will require it to be so.

The interval G inherits the natural linear order $\leq$ of the real numbers $\mathbf{R}$. Since G is a nonempty open interval of $\mathbf{R}$, each Dedekind section of G determines one and only one element. Suppose $a, b \in G$ and $a \leq b$. If $x_1 \in G$, we claim that a natural requirement is

$$x_1 * a \leq x_1 * b, \tag{12}$$

no matter which interpretation is given to the group product: inequality (12) reflects necessary properties of change on the set G. To see this, note

that when measures of change appear on the right in the group product and elements of G on the left, x_1 changed by amount a is less than or equal to x_1 changed by amount b and (12) should hold. Conversely, if measures of change appear on the left in the group product and elements of G on the right, then since a and b are each changed by the same amount x_1 and $a \leq b$, (12) should again hold. For $x_1, x_2, a, b \in$ G, and $a \leq b$, a repetition of this same argument indicates that we should also require

$$x_1 * a * x_2 \leq x_1 * b * x_2. \tag{13}$$

Clearly the validity of (13) for all $x_1, x_2 \in G$ also implies $a \leq b$. These considerations of order suggest the following property for the group (G, $*$):

[O] For $a, b \in$ G, $a \leq b$ if and only if $x_1 * a * x_2 \leq x_1 * b * x_2$ for every $x_1, x_2 \in G$.

Property [O] makes (G,$*$) a linearly ordered continuous group. In this context the adjective continuous means that each Dedekind section of the set G determines one and only one element of G. We now recall the following result:

Theorem. Suppose (G,$*$) is a continuous linearly ordered group which is not trivial (i.e., which consists of more than just the identity element). Then (G, $*$) is order isomorphic to $(\mathbf{R}, +)$, the naturally ordered additive group of real numbers; that is, there exists an order preserving isomorphism between (G, $*$) and ($\mathbf{R}$,+).

For further references and proofs of this theorem, see Minassian [4], Fuchs [5], and Loonstra [6]. A different approach, beginning with the functional equation in property [P_4] and leading to essentially the same conclusion, is given by Aczél [7], pp. 273-278. The theorem implies that (G,$*$) is abelian so that *both* interpretations of the binary product $x_1 * x_2$ considered previously are valid: $x_1 * x_2$ is the element of G obtained by changing x_1 by the amount x_2 *or* by changing x_2 by the amount x_1. The theorem also implies the existence of an order preserving isomorphism between (G, $*$) and ($\mathbf{R}$,+), that is, an increasing bijection $f : \mathbf{R} \to G$ satisfying

$$f^{-1}(x_1 * x_2) = f^{-1}(x_1) + f^{-1}(x_2), \tag{14}$$

for all $x_1, x_2 \in G$.

What group products $*$ are possible, and what choice do we have for bijections $f : \mathbf{R} \to G$ satisfying (14)? Since the sets G and $\mathbf{R}$ have the same cardinality, *any* bijection $f : \mathbf{R} \to G$ induces a group structure on G isomorphic to $(\mathbf{R}, +)$ with group product and inverse defined by

$$\begin{aligned} x_1 * x_2 &= f(f^{-1}(x_1) + f^{-1}(x_2)), \\ (x_1)^{-1} &= f(-f^{-1}(x_1)), \end{aligned} \tag{15}$$

for $x_1, x_2 \in G$. To be an *order preserving* group isomorphism, the bijection f must be strictly increasing and hence continuous. The inverse map f^{-1} is strictly increasing and continuous too. Consequently any increasing bijection $f : \mathbf{R} \to G$ induces group operations on G through (15) and, at the same time, becomes an order preserving group isomorphism between the induced group (G, $*$) and ($\mathbf{R}$, +) which is also a homeomorphism.

Clearly there are many possible group products on the set G. Precisely what constitutes a 'natural' group product for G will be examined in the next section. For the moment, let us see how the induced group $(G, *)$ determines an answer to question [Q]. We *define* the change in variable x from x_1 to x_2 and the change in variable y from y_1 to y_2 to be equivalent if they are equal:

$$x_2/x_1 = y_2/y_1. \tag{16}$$

Equation (9) allows (16) to be written in terms of the induced group operations:

$$x_2 * x_1^{-1} = y_2 * y_1^{-1}. \tag{17}$$

The notion of equivalent change in (16) and (17) generates an equivalence relation $\sim$ on $G \times G$, the set of ordered pairs of elements of G. We write $(x_1, x_2) \sim (y_1, y_2)$ if, and only if, (16) and (17) are satisfied. Now suppose we are given x_1, x_2, and $y_1 \in$ G. The value y_2 of y answering question [Q] must make $(y_1, y_2) \sim (x_1, x_2)$, or $x_2 * x_1^{-1} = y_2 * y_1^{-1}$. Solving for y_2 then gives

$$y_2 = x_2 * x_1^{-1} * y_1 \tag{18}$$

as the answer to question [Q]. Since there are many possible group products on G, there are many possible measures of change (9), and many possible answers to question [Q] of the form (18).

3. Unique Answers to Question [Q]. For a given open interval G, any increasing bijection $f : \mathbf{R} \to G$ induces, through (15), a group structure on G order isomorphic to the additive group of real numbers ($\mathbf{R}$,+). Though essentially only one group structure for G exists, individual bijections may induce different group products which determine numerically different measures of change on G and numerically different answers to question [Q]. It is possible, however, for two increasing bijections and the group operations they induce on G to determine the *same* answer to question [Q]. Two groups, $(G, \circ)$ and $(G, *)$, will be called *equivalent* if they determine the same answer to question [Q]. We shall denote equivalence of these groups by $(G, \circ) \equiv (G, *)$. By (18), $(G, \circ) \equiv (G, *)$ if and only if

$$x_2 \circ (x_1)_\circ^{-1} \circ y_1 = x_2 * x_1^{-1} * y_1 \tag{19}$$

for all $x_1, x_2, y_1 \in$ G, where $\circ$ and $(\cdot)_\circ^{-1}$ are the group operations of $(G, \circ)$ and $*$ and $(\cdot)^{-1}$ are the group operations of (G,$*$). Since the results of section 2 imply that the group structure appropriate for measuring change on G and answering question [Q] is order isomorphic to $(\mathbf{R}, +)$, the equivalence relation $\equiv$ partitions the set of all group structures on G order isomorphic to ($\mathbf{R}$,+) into disjoint equivalence classes. Two groups belong to the same equivalence class if and only if both yield the same answer to question [Q].

Suppose $e_\circ \in G$ is the identity element of $(G, \circ)$ and $(G, \circ) \equiv (G, *)$. These groups are both order isomorphic to $(\mathbf{R}, +)$, so they are isomorphic to each other. Since they are equivalent, though, the relationship between them takes a special form. If we put $x_1 = e_\circ$ in (19), then

$$x_2 \circ y_1 = x_2 * y_1 * e_\circ^{-1} \tag{20}$$

for $x_2, y_1 \in$ G. (Note that $e_\circ^{-1}$ in (20) is the inverse of $e_\circ$ *with respect to the group* $(G, *)$.) If we put $x_2 = e_\circ$ and $y_1 = e_\circ$ in (19), we conclude that

$$(x_1)_\circ^{-1} = x_1^{-1} * e_\circ * e_\circ \tag{21}$$

for $x_1 \in G$. Conversely, the group $(G, \circ)$ defined by the operations in (20) and (21) has $e_\circ \in G$ as its identity element and is equivalent to the group $(G, *)$. It follows that the equivalence class of the group (G, $*$) with respect to the equivalence relation $\equiv$ is the one-parameter family of groups

$\{(G, *_{e_o}) : e_o \in G\}$ whose members's group operations are defined by

$$\begin{aligned} x_1 *_{e_o} x_2 &= x_1 * x_2 * e_o^{-1}, \\ (x_1)_{e_o}^{-1} &= x_1^{-1} * e_o * e_o. \end{aligned} \tag{22}$$

The parameter $e_o \in G$ indexing the family $\{(G, *_{e_o}) : e_o \in G\}$ is the identity element of the group $(G, *_{e_o})$ and determines its group operations through (22). Every member of this one-parameter family of groups necessarily determines the same answer to question [Q]—that given in (18). Equation (22) implies that the equivalent groups $(G, *_{e_o})$ and $(G, *)$ are isomorphic with isomorphism

$$f_{e_o} : G \to G;\ f_{e_o}(x) = x * e_o, \tag{23}$$

so that

$$\begin{aligned} x_1 *_{e_o} x_2 &= f_{e_o}(f_{e_o}^{-1}(x_1) * f_{e_o}^{-1}(x_2)) = x_1 * x_2 * e_o^{-1}, \\ (x_1)_{e_o}^{-1} &= f_{e_o}([f_{e_o}^{-1}(x_1)]^{-1}) = x_1^{-1} * e_o * e_o, \end{aligned} \tag{24}$$

for $x_1, x_2 \in$ G. When e_o is also the identity element of the group $(G, *)$ (23) and (24) imply that the map $f_{e_o} : G \to G$ is the identity and the groups $(G, *_{e_o})$ and $(G, *)$ are identical.

What distinguishes the familiar answers to question [Q] for G = $\mathbf{R}$ in (3) and G = $\mathbf{R}^{\text{pos}}$ in (4)? To obtain (3), we can take (G,$*$) = ($\mathbf{R}$,+), so that $x_1 * x_2 = x_1 + x_2$ and the measure of change in x from x_1 to x_2 is the difference $x_2 * x_1^{-1} = x_2 - x_1$. Thus equivalent change is synonomous with equal differences, as in (1). To obtain (4), we can take (G,$*$) = ($\mathbf{R}^{\text{pos}}, \cdot$), so $x_1 * x_2 = x_1 \cdot x_2$ and the measure of change in x from x_1 to x_2 is the ratio or proportion $x_2 * x_1^{-1} = \frac{x_2}{x_1}$. Thus, equivalent change is synonomous with equal proportions, as in (2). To see what is special about these group products, measures of change, and answers to question [Q], consider G = $\mathbf{R}$, for example, and the increasing bijection $f : \mathbf{R} \to G;\ x \mapsto x^3$. The group $(G, *)$ induced on G by the bijection f has group operations given by (15):

$$\begin{aligned} x_1 * x_2 &= (x_1^{1/3} + x_2^{1/3})^3, \\ x_1^{-1} &= -x_1, \end{aligned} \tag{25}$$

for $x_1, x_2 \in G$. The corresponding measure of change and answer to question [Q] determined by the group (G, $*$) are given by (9) and (18) respectively:

$$\begin{aligned} x_2 * x_1^{-1} &= [x_2^{1/3} - x_1^{1/3}]^3, \\ y_2 &= x_2 * x_1^{-1} * y_1 = \left[x_2^{1/3} - x_1^{1/3} + y_1^{1/3}\right]^3. \end{aligned} \tag{26}$$

What is most evident about the group product in (25) and the measure of change and answer to question [Q] given in (26) compared to those determined by $(\mathbf{R}, +)$ is their lack of *rational dependence* on x_1, x_2, and y_1. It is rational dependence of the group product on its factors which gives a particularly simple form for the new element $x_1 * x_2$ obtained by changing the element x_1, say, by the amount x_2. (We shall see that this rational dependence also carries over to the measure of change and the answer to question [Q] determined by $(G, *)$.)

For a given interval G, what, then, is the most general increasing bijection $f : \mathbf{R} \to G$ whose induced group product $*$ defined on G by (15) depends *rationally* on its factors ? We would like

$$x_1 * x_2 = R(x_1, x_2), \tag{27}$$

where R is a rational function of $x_1, x_2 \in G$. If we combine (27) with (15), this means we are seeking increasing bijections $f : \mathbf{R} \to G$ which satisfy the equation

$$f[f^{-1}(x_1) + f^{-1}(x_2)] = R(x_1, x_2), \tag{28}$$

for $x_1, x_2 \in G$. Let $x_1 = f(u_1)$ and $x_2 = f(u_2)$ for $u_1, u_2 \in \mathbf{R}$. Then (28) implies that any increasing bijection $f : \mathbf{R} \to G$ which induces a rational group product on G must satisfy the equation

$$f(u_1 + u_2) = R(f(u_1), f(u_2)), \tag{29}$$

for all $u_1, u_2 \in \mathbf{R}$ and some rational function R. Simply put, the increasing bijections we seek, necessarily continuous, must satisfy a *rational* addition theorem.

L.E. Dickson [8], W. Alt [9], and A. Kuwagaki [10] have shown that the only continuous functions satisfying a rational addition theorem on an interval are those of the form

$$f_\alpha(x) = \frac{Ax + B}{Cx + D}, \qquad f_\beta(x) = \frac{Ae^{cx} + B}{Ce^{cx} + D}, \tag{30}$$

where A,B,C,D, and c are arbitrary (perhaps complex) constants; see Aczél [7], p. 61. Hence, if $f : \mathbf{R} \to G$ is an increasing bijection with one of the forms in (30), the resulting induced group $(G, *)$ is order isomorphic to $(\mathbf{R}, +)$ and the induced group product $*$ in (15) depends rationally on its factors. It is the open interval G which determines the appropriate map in (30).

For G = $\mathbf{R}$ an increasing bijection from $\mathbf{R}$ to G of the form (30) must be an affine map of the form

$$f_\alpha : \mathbf{R} \to G;\ x \mapsto Ax + B,\ \ (A > 0, B \in \mathbf{R}). \tag{31}$$

For $G = \mathbf{R}^{\text{pos}}$, the map

$$f_{\beta 1} : \mathbf{R} \to G; x \mapsto e^{Ax+B},\ \ (A > 0, B \in \mathbf{R}), \tag{32}$$

is the only increasing bijection from $\mathbf{R}$ to G of the form (30). Finally, the map

$$f_{\beta 2} : \mathbf{R} \to G;\ x \mapsto \frac{e^{Ax+B}}{1 + e^{Ax+B}},\ \ (A > 0, B \in \mathbf{R}) \tag{33}$$

is the only increasing bijection from $\mathbf{R}$ to G of the form (30) for $G = (0, 1)$. The *two-parameter families* of maps defined by (31), (32), and (33) for $A > 0$ and $B \in \mathbf{R}$ are related. For each $A > 0, B \in \mathbf{R}$, the maps $f_{\beta 1}$ and $f_{\beta 2}$ are determined from f_α by

$$\begin{aligned} f_{\beta 1} &= \exp \circ f_\alpha, \\ f_{\beta 2} &= r \circ f_{\beta 1} = r \circ \exp \circ f_\alpha, \end{aligned} \tag{34}$$

where $\exp : \mathbf{R} \to \mathbf{R}^{\text{pos}}; x \mapsto e^x$ is the exponential function and

$$r : \mathbf{R}^{\text{pos}} \to (0, 1);\ x \mapsto \frac{x}{1 + x} \tag{35}$$

is a linear fractional transformation.

To investigate the group operations, measures of change, and answers to question [Q] determined by these families of increasing bijections, we first consider G = $\mathbf{R}$. Given $A > 0$ and $B \in \mathbf{R}$, the group operations induced on G by a map in (31) are

$$\begin{aligned} x_1 * x_2 &= f_\alpha(f_\alpha^{-1}(x_1) + f_\alpha^{-1}(x_2)) = x_1 + x_2 - B, \\ x_1^{-1} &= f_\alpha(-f_\alpha^{-1}(x_1)) = -x_1 + 2B, \end{aligned} \tag{36}$$

for $x_1, x_2 \in G$. Note that the group operations defined in (36) depend on the parameter B only. We shall denote the one-parameter family of groups on G determined by the operations in (36) by $\{(G, *_B) : B \in \mathbf{R}\}$. The group operations in (36) associated with $(G, *_B)$ will be denoted by $*_B$ and $(\cdot)_B^{-1}$, where

$$\begin{aligned} x_1 *_B x_2 &= x_1 + x_2 - B, \\ (x_1)_B^{-1} &= -x_1 + 2B, \end{aligned} \tag{37}$$

for $x_1, x_2 \in G$ and $B \in \mathbf{R}$. The parameter $B \in \mathbf{R}$ used to index the family $\{(G, *_B) : B \in \mathbf{R}\}$ is the identity element of the group $(G, *_B)$. The group operations in (37) result from (22) if in those equations we take $e_\circ = B$ and $(G, *) = (\mathbf{R}, +)$. Hence, when $G = \mathbf{R}$ and $B \in \mathbf{R}$, the groups $(\mathbf{R}, +)$ and $(G, *_B)$ are equivalent, $(\mathbf{R}, +) \equiv (G, *_B)$, and the equivalence class of $(\mathbf{R},+)$ is the one-parameter family $\{(G, *_B) : B \in \mathbf{R}\}$. The measure of change determined by (9) and the group $(G, *_B)$ is

$$x_2 *_B (x_1)_B^{-1} = x_2 - x_1 + B, \tag{38}$$

a translate of the difference $x_2 - x_1$. Finally, from (18), the answer to question [Q] common to each of the groups $(G, *_B), B \in \mathbf{R}$, is

$$y_2 = x_2 *_B (x_1)_B^{-1} *_B y_1 = (x_2 - x_1 + B) + y_1 - B = x_2 - x_1 + y_1. \tag{39}$$

This is the 'equal differences' result in (3). We conclude that the 'equal differences' value of y_2 in (3) is the appropriate *unique* answer to question [Q] when G = $\mathbf{R}$ and the group used to determine a measure change on G, necessarily order isomorphic to $(\mathbf{R}, +)$, has a group product depending *rationally* on its factors.

When G = $\mathbf{R}^{\text{pos}}$, $A > 0$, and $B \in \mathbf{R}$, a map of the form (32) induces the following group operations on G:

$$\begin{aligned} x_1 * x_2 &= h_{\beta 1}(h_{\beta 1}^{-1}(x_1) + h_{\beta 1}^{-1}(x_2)) = \frac{x_1 x_2}{e^B}, \\ x_1^{-1} &= h_{\beta 1}(-h_{\beta 1}^{-1}(x_1)) = \frac{e^{2B}}{x_1}. \end{aligned} \tag{40}$$

Again, the group operations in (40) depend only on the parameter $B \in \mathbf{R}$, or equivalently the element $e^B \in \mathbf{R}^{\text{pos}}$. The one-parameter family of groups on $\mathbf{R}^{\text{pos}}$ determined by the operations in (40) will be denoted by $\left\{(G, *_{e^B}) : e^B \in \mathbf{R}^{\text{pos}}, B \in \mathbf{R}\right\}$. The group operations in (40) associated with $(G, *_{e^B})$ will be denoted by $*_{e^B}$ and $(\cdot)^{-1}_{e^B}$, where

$$\begin{aligned} x_1 *_{e^B} x_2 &= \frac{x_1 x_2}{e^B}, \\ (x_1)^{-1}_{e^B} &= \frac{e^{2B}}{x_1} \end{aligned} \tag{41}$$

for $x_1, x_2 \in G$ and $e^B \in G$. The identity element e^B of the group $(G, *_{e^B})$ has been chosen as the parameter indexing this one-parameter family. Again, the group operations in (41) result from (22) if in those equations we take $e_\circ = e^B$, and $(G, *) = (\mathbf{R}^{\text{pos}}, \cdot)$, the multiplicative group of positive real numbers. Hence the groups $(\mathbf{R}^{\text{pos}}, \cdot)$ and $(G, *_{e^B})$ are equivalent, $(\mathbf{R}^{\text{pos}}, \cdot) \equiv (G, *_{e^B})$ for all $e^B \in G$, $B \in \mathbf{R}$. When $G = \mathbf{R}^{\text{pos}}$ the equivalence class of the group $(\mathbf{R}^{\text{pos}}, \cdot)$ is the one-parameter family of groups $\left\{(G, *_{e^B}) : e^B \in \mathbf{R}^{\text{pos}}, B \in \mathbf{R}\right\}$. The measure of change determined by (9) and the group $(G, *_{e^B})$ is

$$x_2 *_{e^B} (x_1)^{-1}_{e^B} = \frac{e^B x_2}{x_1}, \tag{42}$$

a dilatation of the proportion $\frac{x_2}{x_1}$. The answer to question [Q] in (18), common to each of the groups $(G, *_{e^B}), e^B \in G$, is

$$y_2 = x_2 *_{e^B} (x_1)^{-1}_{e^B} *_{e^B} y_1 = \frac{e^B x_2}{x_1} \frac{y_1}{e^B} = \frac{x_2}{x_1} y_1. \tag{43}$$

This is the 'equal proportions' result in (4) . We conclude that the 'equal proportions' value of y_2 in (4) is the appropriate *unique* answer to question [Q] when $G = \mathbf{R}^{\text{pos}}$ and the group used to determine a measure change on G, necessarily order isomorphic to $(\mathbf{R}, +)$, has a group product depending *rationally* on its factors.

When $G = (0, 1)$ the group operations induced on G by a map in (33) are

$$x_1 * x_2 = h_{\beta 2}(h^{-1}_{\beta 2}(x_1) + h^{-1}_{\beta 2}(x_2)) = \frac{x_1 x_2}{x_1 x_2 + (1 - x_1)(1 - x_2) e^B},$$

(44)

$$x_1^{-1} = h_{\beta 2}(-h_{\beta 2}^{-1}(x_1)) = \frac{1 - x_1}{1 + (e^{-2B} - 1)x_1}.$$

The one-parameter family of groups on (0,1) determined by the operations in (44) will be denoted by $\left\{(G, *_{r(e^B)}) : r(e^B) \in (0,1), B \in \mathbf{R}\right\}$, where

$$r(e^B) = \frac{e^B}{1 + e^B} \tag{45}$$

is the identity element of $(G, *_{r(e^B)})$ and the map $r : \mathbf{R}^{\text{pos}} \to (0,1)$ is defined in (35). The group operations in (44) associated with $(G, *_{r(e^B)})$ will be denoted by $*_{r(e^B)}$ and $(\cdot)^{-1}_{r(e^B)}$, where

$$\begin{aligned} x_1 *_{r(e^B)} x_2 &= \frac{x_1 x_2}{x_1 x_2 + (1 - x_1)(1 - x_2)e^B}, \\ (x_1)^{-1}_{r(e^B)} &= \frac{1 - x_1}{1 + (e^{-2B} - 1)x_1}, \end{aligned} \tag{46}$$

for $x_1, x_2 \in G$ and $B \in \mathbf{R}$. They result from (22) if in those equations we set $e_o = r(e^B)$, and define the group $(G, *)$ by

$$\begin{aligned} x_1 * x_2 &= \frac{x_1 x_2}{x_1 x_2 + (1 - x_1)(1 - x_2)}, \\ (x_1)^{-1} &= 1 - x_1 \end{aligned} \tag{47}$$

for $x_1, x_2 \in G$. Thus for $B \in \mathbf{R}$ the groups $(G, *_{r(e^B)})$, $r(e^B) \in G$, and $(G, *)$ are equivalent, $(G, *_{r(e^B)}) \equiv (G, *)$, and when B = 0 the two groups are identical. For $G = (0,1)$ the equivalence class of the group $(G, *)$ defined by (47) is the one-parameter family of groups $\left\{(G, *_{r(e^B)}) : r(e^B) \in (0,1), B \in \mathbf{R}\right\}$.

The measure of change determined by $(G, *_{r(e^B)}), r(e^B) \in G$, and (9) is

$$x_2 *_{r(e^B)} (x_1)^{-1}_{r(e^B)} = \frac{x_2(1 - x_1)}{x_2(1 - x_1) + x_1(1 - x_2)e^{-B}}, \tag{48}$$

and the answer to question [Q] common to each member of this one-parameter family of groups is, from (18),

$$y_2 = x_2 *_{r(e^B)} (x_1)^{-1}_{r(e^B)} *_{r(e^B)} y_1 = \frac{x_2 y_1(1 - x_1)}{x_1(1 - x_2) + y_1(x_2 - x_1)}. \tag{49}$$

This is the result in (5). We conclude that the value of y_2 in (5) given by Ng [1] is the appropriate *unique* answer to question [Q] when G = (0,1) and the group structure used to determine a measure of change on G, necessarily order isomorphic to $(\mathbf{R}, +)$, has a group product depending *rationally* on its factors.

4. Properties leading to (3), (4), and (5). In this section we summarize sections 2 and 3 by formulating a set of properties for determining equivalent change on G which leads to a unique answer to question [Q]. We also give geometrical interpretations of the answers to question [Q] in (3), (4), and (5) for $G = \mathbf{R}, \mathbf{R}^{\text{pos}}$, and $(0, 1)$ respectively.

Given an open interval G of $\mathbf{R}$, we assume that for every ordered pair (x_1, x_2) of elements of G there is a unique element of G which represents the change in variable x from x_1 to x_2 (Assumption [A]). We assume that a measure of change exists on G, the binary operation /, which satisfies properties $[P_0] - [P_4]$ of section 2, and, upon considering the notion of order, that the resulting group $(G, *)$ satisfies Property [O]. Then the theorem of section 2 implies that the group $(G, *)$ is order isomorphic to $(\mathbf{R}, +)$. Now any increasing bijection $f : \mathbf{R} \to G$ induces such a group structure on G through the group operations in (15), and different bijections induce different measures of change (9) and different answers (18) to question [Q]. Motivated by (1) and (2), we call *simple* any group product on G which depends *rationally* on its factors. Restricting attention to simple group products narrows the class of possible bijections down to those given in (30). Some calculation then implies that when $G = \mathbf{R}, \mathbf{R}^{\text{pos}}$, and (0,1) and the group product in $(G, *)$ is simple, (3), (4), and (5) respectively arise from (18) as the *unique* answers to question [Q].

Figures 1, 2, and 3 give geometrical interpretations of the answers to question [Q] in (3), (4), and (5). Each figure shows two copies of the set G, G_x for variable x and G_y for variable y. Given $x_1, x_2 \in G_x$, and $y_1 \in G_y$, to determine the answer to question [Q] we begin at $y_1 \in G_y$ and use the map

$$T : G_y \to G_x;\ y \mapsto y * x_1 * y_1^{-1} \tag{50}$$

to 'transfer' to $x_1 \in G_x : T(y_1) = x_1$. Once at $x_1 \in G_x$, the map

$$C : G_x \to G_x;\ x \mapsto x * x_2 * x_1^{-1} \tag{51}$$

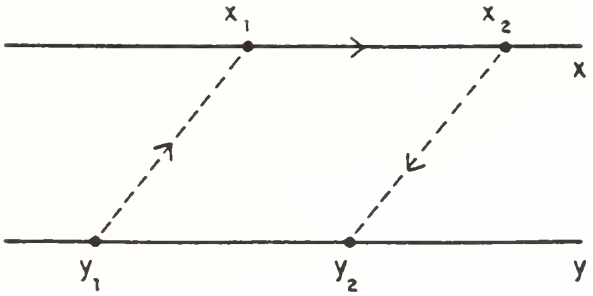

Fig. 1

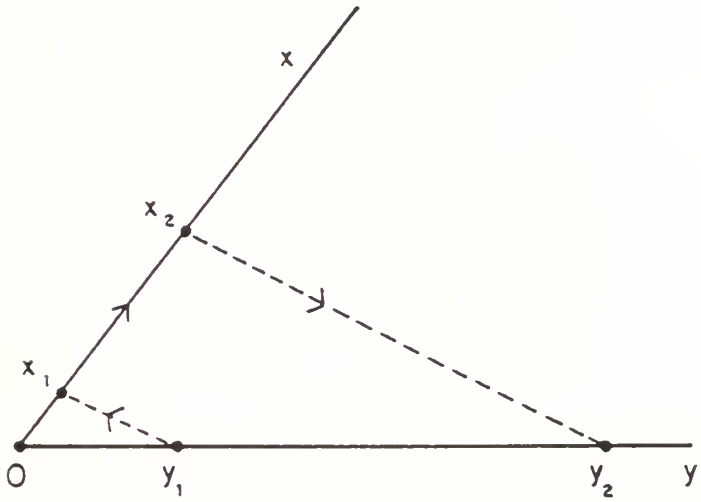

Fig. 2

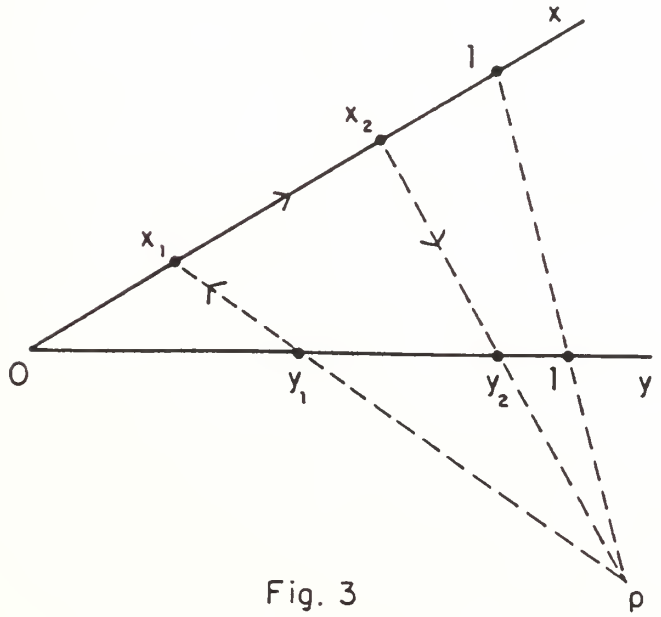

Fig. 3

'changes' x_1 by the amount $x_2 * x_1^{-1}$ and moves us to $x_2 \in G_x$: $C(x_1) = x_2$. Next we use the inverse of the map in (50),

$$T^{-1} : G \to G;\ y \mapsto y * x_1^{-1} * y_1, \tag{52}$$

to 'transfer' back to $y_2 \in G_y$:

$$y_2 = T^{-1}(x_2) = x_2 * x_1^{-1} * y_1. \tag{53}$$

This is the answer to question [Q] in (18). If we combine (50), (51), and (52), the answer to question [Q] on G is given by $y_2 = T^{-1} \circ C \circ T(y_1)$, where the map

$$T^{-1} \circ C \circ T : G_y \to G_y;\ y \mapsto x_2 * x_1^{-1} * y \tag{54}$$

'changes' variable $y \in G_y$ by $x_2 * x_1^{-1}$, the change occuring in variable $x \in G_x$. When $G = \mathbf{R}$, from (39), (50), (51), and (54) we have

$$\begin{aligned} T : G_y \to G_x;\ x &\mapsto x + (x_1 - y_1), \\ C : G_x \to G_x;\ x &\mapsto x + (x_2 - x_1), \\ T^{-1} \circ C \circ T : G_y \to G_y;\ y &\mapsto y + x_2 - x_1, \end{aligned} \tag{55}$$

so that the maps in (50), (51), and (54) are translations. When $G = \mathbf{R}^{\text{pos}}$, (43), (50), (51), and (54) give

$$\begin{aligned} T : G_y \to G_x;\ x &\mapsto x\frac{x_1}{y_1}, \\ C : G_x \to G_x;\ x &\mapsto x\frac{x_2}{x_1}, \\ T^{-1} \circ C \circ T : G_y \to G_y;\ y &\mapsto y\frac{x_2}{x_1}, \end{aligned} \tag{56}$$

so the maps in (50), (51), and (54) are dilatations. Finally, for $G = (0, 1)$, from (49), (50), (51), and (54) we have

$$\begin{aligned} T : G_y \to G_x;\ x &\mapsto \frac{x(x_1)(1 - y_1)}{y_1(1 - x_1) + x(x_1 - y_1)}, \\ C : G_x \to G_x;\ x &\mapsto \frac{x(x_2)(1 - x_1)}{x_1(1 - x_2) + x(x_2 - x_1)}, \\ T^{-1} \circ C \circ T : G_y \to G_y;\ y &\mapsto \frac{y(x_2)(1 - x_1)}{x_1(1 - x_2) + y(x_2 - x_1)}, \end{aligned} \tag{57}$$

and the maps in (50), (51), and (54) are linear fractional transformations. Figure 3 and (57) imply that the answer to question [Q] in (5) is geometrically determined by projection from a point P. The points 0 and 1 in G_x and G_y are fixed under projection from P, and P itself is determined by the intersection of the lines through $x_1 \in G_x, y_1 \in G_y$ and $1 \in G_x, 1 \in G_y$ (the acute angle between the two copies of G affects the location of P but does not affect the answer y_2). Once P is determined, the answer $y_2 \in G_y$ is obtained as the projection of $x_2 \in G_x$ from P to G_y. This geometrical interpretation implies that the cross-ratios of the points $\{0, x_1, x_2, 1\}$ and $\{0, y_1, y_2, 1\}$ are equal, that is

$$R(0, x_1; x_2, 1) = R(0, y_1; y_2, 1), \tag{58}$$

where

$$R(x, y; z, t) = \frac{z - x}{z - y} / \frac{t - x}{t - y}; \tag{59}$$

see Burn [8, p. 43], for example. A combination of (58) and (59) yields the value for y_2 given in (5).

Appendix. In [1], variables x and y belonged to the closed unit interval [0,1], and the answer to question [Q] was assumed to have the form

$$y_2 = F(x_1, y_1, x_2), \tag{60}$$

where $F : [0,1]^3 \to [0,1]$ is a map from the closed unit cube $[0,1]^3$ to [0,1]. Ng gave twelve 'reasonable' properties he felt the map F should have, and showed that the answer in (5) has all of them. We have seen in sections 2, 3, and 4 how the answer to question [Q] in (5) is determined. In this appendix we shall examine the extent to which Ng's twelve properties constrain possible answers to question [Q] on $G = (0,1)$. The results of section 2 imply that measurement of change on G requires a group structure $(G, *)$ order isomorphic to $(\mathbf{R}, +)$. e will assume that the answer to question [Q] is given by a combination of the group operations in (15) and (18):

$$\begin{aligned} y_2 &= x_2 * x_1^{-1} * y_1 \\ &= f(f^{-1}(x_2) - f^{-1}(x_1) + f^{-1}(y_1)) \\ &= \bar{F}(x_1, y_1, x_2), \end{aligned} \tag{61}$$

where $f : \mathbf{R} \to G$ is the increasing bijection inducing the group structure $(G, *)$ on G and the map $\bar{F} : G^3 \to G$ is defined in terms of the bijection f by the third equality in (61). What constraints do Ng's twelve properties put on the map $\bar{F}$ in (61), and more specifically, what constraints do they put on the increasing bijection $f : \mathbf{R} \to G$ on which $\bar{F}$ depends? We shall see that only one of Ng's properties, Property 9, puts a real constraint on the increasing bijection $f : \mathbf{R} \to G$.

Before we compare the properties postulated by Ng for the map in (60) with those of the map $\bar{F}$ in (61), we must reconcile a difference in their domains. In Ng [1], the domain and co-domain of the map F in (60) are defined to be the closed unit cube $[0,1]^3$ and the closed unit interval [0,1] respectively. In (61) the domain and co-domain of the map $\bar{F}$ are the open unit cube G^3 and the open unit interval G, where $G = (0, 1)$. Now the function proposed by Ng in (5) is *not* defined everywhere on $[0,1]^3$ (the points $(0,0,x_2)$ and $(1,y_1,1)$, for example, do not belong to its domain), so it is clear that the domain of F needs modification. Since $f : \mathbf{R} \to G$ is an increasing bijection, the extended real numbers $-\infty$ and ∞ (the greatest lower and least upper bounds of $\mathbf{R}$) correspond to the numbers 0 and 1 (the greatest lower and least upper bounds of G) respectively. The numbers 0 and 1 are also the greatest lower and least upper bounds for measures of change on G, and play roles for G analogous to the roles played by the extended real numbers $-\infty$ and ∞ for $\mathbf{R}$. They cannot be included in the group $(G, *)$ in exactly the same way that the extended real numbers $-\infty$ and ∞ cannot be included in the additive group $(\mathbf{R}, +)$. We therefore modify Ng's properties appropriately so that the domain and co-domain of the map F in (60) agree with those of the map $\bar{F}$ in (61). Ng's first two properties are

> PROPERTY 1 (COMPLETENESS). F$(\cdot)$ exists for all x_1, y_1, and x_2 between and inclusive of zero and one.
>
> PROPERTY 2 (UNIQUENESS). For each and every value of (x_1, y_1, x_2), $F(\cdot)$ is unique.

We change 'inclusive' in Property 1 to 'exclusive' as mentioned above. Property 2 is satisfied by (60) and (61) since both are maps.

Recall from section 2 that we defined an equivalence relation $\sim$ on $G \times G$, the set of ordered pairs of G:

$$(x_1, x_2) \sim (y_1, y_2) \text{ if and only if } x_2 * x_1^{-1} = y_2 * y_1^{-1}. \tag{62}$$

Clearly $(x_1, x_2) \sim (y_1, y_2)$ if and only if y_2 is defined by the map $\bar{F}$ in (61). A second relation $\approx$ on $G \times G$ can be defined by the map F in (60):

$$(x_1, x_2) \approx (y_1, y_2) \text{ if and only if } y_2 = F(x_1, y_1, x_2). \tag{63}$$

Ng's next three properties are

PROPERTY 3 (INTERCHANGEABILITY).
$F(x_2,F(x_1,y_1,x_2),x_1) = y_1$.

PROPERTY 4 (PARITY). $F(x_1,y_1, x_2) = x_2$ for all $x_1 = y_1$ and all x_2.

PROPERTY 5 (IDENTITY). $F(x_1,y_1, x_2) = y_1$ if $x_2 = x_1$.

Property 3 holds for the map $\bar{F}$ in (61), since $y_2 = \bar{F}(x_1, y_1, x_2)$ iff $(x_1, x_2) \sim (y_1, y_2)$ iff $x_2 * x_1^{-1} = y_2 * y_1^{-1}$ iff $x_1 * x_2^{-1} = y_1 * y_2^{-1}$ iff $(x_2, x_1) \sim (y_2, y_1)$ iff $y_1 = \bar{F}(x_2, y_2, x_1)$ (where iff stands for 'if and only if'). Properties 4 and 5 also hold for the map $\bar{F}$ in (61). Property 4 states that when x and y have the same initial value, $x_1 = y_1$, the final value y_2 of y giving a change in y equivalent to the change in x must be equal to the final value x_2 of x. Property 5 states that if variable x remains unchanged, variable y also remains unchanged.

The next property considered by Ng is

PROPERTY 6 (LIMITATION). $F(x_1, y_1, 0) = 0$; $F(x_1, y_1, 1) = 1$.

We modify Property 6 so that the domain of the map F in (60) agrees with the domain of the map $\bar{F}$ in (61):

PROPERTY 6′ (LIMITATION). For $x_1, y_1 \in G$,

$$\lim_{x_2 \to 0^+} F(x_1, y_1, x_2) = 0 \ , \quad \lim_{x_2 \to 1^-} F(x_1, y_1, x_2) = 1.$$

Property 6′ is satisfied by the map $\bar{F}$ in (61) because, for fixed $x_1, y_1 \in G$, $f^{-1}(x_2) \to -\infty$ as $x_2 \to 0^+$ and $f^{-1}(x_2) \to \infty$ as $x_2 \to 1^-$ since $f : \mathbf{R} \to G$ is an increasing bijection. This property illustrates the role of 0 and 1 as the greatest lower and least upper bound respectively for change on G; if $x_1 \in G$, and $x_2 \to 0^+$, variable x undergoes the largest decrease possible. For variable y, starting from $y_1 \in G$, to undergo an equivalent change we must have $y_2 \to 0^+$ too. Similarly, if $x_2 \to 1^-$, we must have $y_2 \to 1^-$.

> PROPERTY 7 (MONOTONICITY). (i) $F(x_1, y_1, x_2)$ montonically increases in x_2; (ii) $F(x_1, y_1, x_2)$ monotonically increases in y_1 and decreases in x_1 except when it has already reached its limiting points (e.g., when $y_2 = x_2 = 1$).

Disregarding the qualification in part ii) of Property 7 which has been discussed previously, we note that this property is also satisfied by the map $\bar{F}$ in (61) since $f : \mathbf{R} \to G$ is a monotone increasing bijection.

For fixed $x_1, y_1 \in G$, Ng defined the function $f^i : G \to G$ by $f^i(x) = F(x_1, y_1, x)$ His next property discusses the smoothness of the function f^i:

> PROPERTY 8 (DIFFERENTIABILITY). Each and every function $f^i(x)$ is continuous and differentiable at all points [of its domain].

From (61), the map $\bar{f}^i : G \to G$ is defined for $x_1, y_1 \in G$ by

$$\bar{f}^i(x) = f(f^{-1}(x) - f^{-1}(x_1) + f^{-1}(y_1)), \tag{64}$$

for $x \in G$. It is continuous since $f : \mathbf{R} \to G$ is a homeomorphism, and its differentiability can be ensured by requiring f to be a diffeomorphism. Note that when the group product $*$ induced on G by the bijection f depends rationally on its factors, the function $\bar{f}^i$ in (64) is given by (5) with $x_2 = x$ and is a diffeomorphism.

Ng's first eight properties require little of the increasing bijection $f : \mathbf{R} \to G$ defining the map $\bar{F}$ in (61). The next property is the first, and only property of Ng's to impose a substantial constraint on f.

> PROPERTY 9 (COMPLEMENTARITY).
> $F(1 - x_1, 1 - y_1, 1 - x_2) = 1 - F(x_1, y_1, x_2)$.

If x is the probability/percentage of event X, then $1 - x$ is the probability/percentage of non–X. Similarly for y. To understand Property 9, consider the following example given in Ng [1]. Suppose the male employment rate changes from 90% to 99% and we believe that the female employment rate, starting at 70%, must increase to, say, 77% to undergo an equivalent change. Then we must also accept that when the male *unemployment* rate changes from 10% to 1%, the equivalent change for female unemployment, starting from 30%, must be to decrease to 23%.

If we use (61) to express Property 9 for the map $\bar{F}$ in terms of group operations, we must require

$$(1 - x_2) * (1 - x_1)^{-1} * (1 - y_1) = 1 - (x_2 * x_1^{-1} * y_1), \tag{65}$$

for $x_1, x_2, y_1 \in G$. For $a \in G$, set $x_1 = \frac{1}{2}, y_1 = a$, and $x_2 = 1 - a$ in (65) to obtain

$$a * \tfrac{1}{2}^{-1} * (1 - a) = 1 - [(1 - a) * \tfrac{1}{2}^{-1} * a]. \tag{66}$$

Since $(G, *)$ is abelian (66) implies

$$a * (1 - a) = \tfrac{1}{2} * \tfrac{1}{2} \tag{67}$$

for $a \in G$. When expressed in terms of the increasing bijection $f : \mathbf{R} \to G$, (67) takes the form

$$f\left(f^{-1}(a) + f^{-1}(1 - a)\right) = f\left(2f^{-1}\left(\tfrac{1}{2}\right)\right), \tag{68}$$

for $a \in G$. If we put $a = \frac{1}{2} - x$ for $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, (68) implies that the map

$$h : \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \to \mathbf{R};\ h(x) = f^{-1}\left(x + \tfrac{1}{2}\right) - f^{-1}\left(\tfrac{1}{2}\right) \tag{69}$$

is odd:

$$h(-x) = -h(x) \tag{70}$$

for $x \in (-\frac{1}{2}, \frac{1}{2})$. Since $f : \mathbf{R} \to G$ is an increasing bijection, the map $h : \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbf{R}$ in (69) is also an increasing bijection. Thus if the answer to question [Q] determined by the map $\bar{F}$ in (61) is to satisfy Property 9, the increasing bijection $f : \mathbf{R} \to G$ defining $\bar{F}$ must be given by

$$f(x) = \tfrac{1}{2} + h^{-1}\left(x - f^{-1}\left(\tfrac{1}{2}\right)\right) \tag{71}$$

for $x \in \mathbf{R}$, and its inverse $f^{-1} : G \to \mathbf{R}$ by

$$f^{-1}(x) = f^{-1}\left(\tfrac{1}{2}\right) + h\left(x - \tfrac{1}{2}\right), \tag{72}$$

for $x \in G$, where $h : \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbf{R}$ is an odd, increasing bijection and $f^{-1}\left(\frac{1}{2}\right) \in \mathbf{R}$. Conversely, when $f^{-1}\left(\frac{1}{2}\right) \in \mathbf{R}$ and $h : \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbf{R}$ is an odd, increasing bijection, the increasing bijection $f : \mathbf{R} \to G$ defined in (71) determines, through (61), an answer to question [Q] satisfing Property 9.

We note that the answer to question [Q] in (5) satisfies Property 9, since for $A > 0$ and $B \in \mathbf{R}$ each of the increasing bijections $f_{\beta 2} : \mathbf{R} \to G$ in (33), determines, via (69), the same odd, increasing bijection

$$h : \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \to \mathbf{R};\ x \mapsto h(x) = f_{\beta 2}^{-1}\left(x + \tfrac{1}{2}\right) - f_{\beta 2}^{-1}\left(\tfrac{1}{2}\right) = \log \frac{\frac{1}{2} + x}{\frac{1}{2} - x}. \tag{73}$$

Any odd increasing bijection $h : \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbf{R}$ defines an increasing bijection $f : \mathbf{R} \to G$ in (71) whose resulting map $\bar{F}$ in (61) satisfies Property 9. If this increasing bijection $f : \mathbf{R} \to G$ is not a member of the family of maps in (33), the group product on G defined by (15) will not depend rationally on its factors and the answer to question [Q] it determines in (18) will not be given by (5). For example, take

$$h : \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \to \mathbf{R};\ x \mapsto \log\left(\frac{1 + \tan\frac{\pi}{2}x}{1 - \tan\frac{\pi}{2}x}\right). \tag{74}$$

If we choose $f^{-1}\left(\frac{1}{2}\right) = 0$ in (71), the increasing bijection in (71) determined by (74) is

$$f : \mathbf{R} \to G;\ x \mapsto \tfrac{2}{\pi}\tan^{-1}(e^x). \tag{75}$$

The group operations induced on G by the increasing bijection in (75) are

$$\begin{aligned} x_1 * x_2 &= \tfrac{2}{\pi}\tan^{-1}(\tan\tfrac{\pi}{2}x_1 \tan\tfrac{\pi}{2}x_2), \\ x_1^{-1} &= \tfrac{2}{\pi}\tan^{-1}(\cot\tfrac{\pi}{2}x_1), \end{aligned} \tag{76}$$

for $x_1 \in G$ and the answer to question [Q] satisfying Property 9 is, by (61),

$$y_2 = \bar{F}(x_1, y_1, x_2) = \tfrac{2}{\pi}\tan^{-1}\left(\frac{\tan\frac{\pi}{2}x_2 \tan\frac{\pi}{2}y_1}{\tan\frac{\pi}{2}x_1}\right). \tag{77}$$

The next two properties ensure that the relation $\approx$ defined in (63) by the map F in (60), is symmetric and transitive respectively.

PROPERTY 10 (ANONYMITY).
$F(y_1, x_1, F(x_1, y_1, x_2)) = x_2$.

PROPERTY 11 (TRANSITIVITY). If $y_2 = F(x_1, y_1, x_2)$, and $z_2 = F(x_1, z_1, x_2)$, then $y_2 = F(z_1, y_1, z_2)$.

The map $\bar{F}$ in (61) satisfies them since $y_2 = \bar{F}(x_1, y_1, x_2)$ if, and only if, $(x_1, x_2) \sim (y_1, y_2)$, where $\sim$ is the equivalence relation defined in (62). Since Property 4 implies that the relation $\approx$ is reflexive, Properties 10, 11, and 4 together imply that $\approx$ is an equivalence relation too.

Ng gave one further property to "narrow down the permissible functions defining equivalent changes, preferably to a unique function $y_2 = F(x_1, y_1, x_2)$ or a unique family of functions $y = f^i(x)$". (Ng [1], p. 298) This was

PROPERTY 12 (MONOTONICITY IN $\mathrm{d}y/\mathrm{d}x$). If x_1 is larger/smaller than y_1, then $\mathrm{d}y_2/\mathrm{d}x_2$ monotonically increases/decreases in x_2.

Ng argued that this was a reasonable property for the answer to question [Q], and he showed that (5) has it. However, as the next example shows, Property 12 together with the previous eleven do not suffice to uniquely determine the answer in (5). Consider the increasing bijection given in (75). Through (61) it determines the map $\bar{F}$ given in (77), and this latter map satisfies Ng's first eight properties. Property (9) is also satisfied by $\bar{F}$ since (75) and (69) together determine the odd, increasing bijection $h : \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbf{R}$ given in (74). Properties 10 and 11 are satisfied by the map $\bar{F}$ in (77) too. Direct calculation from (77) yields

$$\frac{d^2 y_2}{dx_2^2} = \frac{\pi K}{\left(\cos^2 \frac{\pi}{2} x_2 + K^2 \sin^2 \frac{\pi}{2} x_2\right)^2} \left[1 - K^2\right] \sin \frac{\pi}{2} x_2 \cos \frac{\pi}{2} x_2, \qquad (78)$$

where

$$K = \frac{\tan \frac{\pi}{2} y_1}{\tan \frac{\pi}{2} x_1}. \qquad (79)$$

Hence if x_1 is larger/smaller than y_1, K in (79) is smaller/larger than one and $d^2 y_2/dx_2^2$ in (78) is greater/less than 0. This implies that dy_2/dx_2 monotonically increases/decreases with respect to x_2. Therefore the map $\bar{F}$ in (77)

satisfies all twelve of Ng's properties, yet is not the answer to question [Q] found in (5).

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